SECTION 11.3 AND 11.4: TAYLOR SERIES

Recall in Section 11.1, we defined and worked with Taylor Polynomials:

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{6} f'''(a)(x-a)^3 + \dots + \frac{1}{n!} f^{(n)}(a)(x-a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

If we combine the notions of Taylor Polynomials and Power Series, we get **Taylor Series**:

$$T(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{6} f'''(a)(x-a)^3 + \dots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Note that the nth partial sums of the Taylor Series, S_n, is the $(n-1)$st degree Taylor polynomial, $p_{n-1}(x)$.

Notice that we didn't write $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$ since there are functions $f(x)$ where the corresponding series $T(x)$ converges for all real numbers but $T(x) = f(x)$ only for $x = a$. (See the #90 in the 11.3 homework.)

In the special case $a = 0$, Taylor Series are often referred to as **Maclaurin Series**.

EXAMPLE 1: Find the Taylor Series for the given function and state the interval of convergence.

1. $f(x) = \sin(x)$ centered at $a = 0$.

   Ans: $T(x) = x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \dfrac{x^7}{5040} + - \dots = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$; converges: $(-\infty, \infty)$.

2. $f(x) = \ln(x)$ centered at $a = 1$.

   Ans: $T(x) = (x-1) - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} - \dfrac{(x-1)^4}{4} + - \dots = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1} (x-1)^k}{k}$; converges: $(0, 2]$.

How do we go about proving that a Taylor Series, $T(x)$ actually adds up to the generating function, $f(x)$?

We use the remainder theorem from Section 11.1: given $f(x)$ and its nth degree Taylor polynomial $p_n(x)$:

$$f(x) - p_n(x) = r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

where c is a real number between x and a. Hence to show $T(x)$ converges to $f(x)$, we need to show $r_n(x) \to 0$ for all x in the interval of convergence. For the case of $f(x) = \sin(x)$, $f^{(n+1)}(x)$ is either $\pm\sin(x)$ or $\pm\cos(x)$. Hence $\left|f^{(n+1)}(c)\right| \leq 1$ for all real numbers c. Hence:

$$|r_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!}$$

Since[1] $\dfrac{|x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$, $r_n(x) \to 0$ for all x so $\sin(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ for all x.

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[1]See our work with the sequence $a_n = \dfrac{3^n}{n!}$ in Section 10.2 ...

Note that we already know $\ln(x)$ adds to the series we derived since it is identical to the series we derived using the geometric sum formula in the previous section. In general, we have the following:

UNIQUENESS OF REPRESENTATION: If $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$, then $c_k = \dfrac{f^{(k)}(a)}{k!}$.

EXAMPLE 2: Use the Taylor Series for $\sin(x)$ to find a power series representation for $\cos(x)$.

Ans: $\cos(x) = D_x[\sin(x)] = D_x\left[\sum_{k=0}^{\infty} \dfrac{(-1)^k x^{2k+1}}{(2k+1)!}\right] = \sum_{k=0}^{\infty} \dfrac{(-1)^k x^{2k}}{(2k)!} = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - + \ldots$

The interval of convergence for $\cos(x) = D_x[\sin(x)]$ is $(-\infty, \infty)$.

EXAMPLE 3: Find the Maclaurin Series for $f(x) = e^x$ and prove the series converges to $f(x)$ for all x.

Ans: $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \ldots = \sum_{k=0}^{\infty} \dfrac{x^k}{k!}$; converges for $(-\infty, \infty)$; $|r_n(x)| \leq \dfrac{e^x |x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$.

EXAMPLE 4: Use the known series for e^x to find series for:

1. $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

   Ans: $\sinh(x) = x + \dfrac{x^3}{6} + \dfrac{x^5}{120} + \dfrac{x^7}{5040} + \ldots = \sum_{k=0}^{\infty} \dfrac{x^{2k+1}}{(2k+1)!}$; converges for $(-\infty, \infty)$

2. $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

   Ans: $\cosh(x) = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{24} + \dfrac{x^6}{720} + \ldots = \sum_{k=0}^{\infty} \dfrac{x^{2k}}{(2k)!}$; converges for $(-\infty, \infty)$

EXAMPLE 5: (VIDEO) Using series for $\sin(x)$, $\cos(x)$, and e^x, find the first few nonzero terms of a series for:

1. $e^{-x}\cos(4x)$ Ans: $e^{-x}\cos(4x) = 1 - x - \dfrac{15}{2}x^2 + \dfrac{47}{6}x^3 - \ldots$

2. $\tan(x)$ Ans: $\tan(x) = \dfrac{\sin(x)}{\cos(x)} = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \ldots$

EXAMPLE 6: Use Maclaurin Series to help you with the following:

1. Find the limit: $\lim_{x\to\infty} x^2\left(e^{-1/x^2} - 1\right)$ Ans: $\lim_{x\to\infty} x^2\left(e^{-1/x^2} - 1\right) = -1$

2. Approximate $\int_0^1 \sin\left(x^3\right)\, dx$ using a partial sum of an infinite series to with an error of at most 0.001.

   Ans: $\int_0^1 \sin\left(x^3\right)\, dx = \sum_{k=0}^{\infty} \dfrac{(-1)^k}{(6k+4)(2k+1)!} \approx 0.2\overline{3}$

THE BINOMIAL SERIES

Recall from Precalculus the Binomial Theorem: for whole numbers n:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\, x^k$$

It turns out the Binomial Theorem is just a special instance of the Maclaurin Series Expansion for $f(x) = (1+x)^p$:

THEOREM (THE BINOMIAL SERIES): For x in $(-1, 1)$:

$$(1+x)^p = \sum_{k=0}^{\infty} \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!}\, x^k = 1 + p\,x + \frac{p\,(p-1)}{2}\, x^2 + \frac{p\,(p-1)\,(p-2)}{3!}\, x^3 + \dots$$

NOTE 1: Depending on the value of p, the series may converge at $x = \pm 1$.

NOTE 2: If $p = n$ is a whole number, the Binomial Series reverts back to the Binomial Theorem!

EXAMPLE 7: Use the Binomial Series to approximate $\sqrt{1.1}$ accurate to within three decimal places.

Ans: $\sqrt{1+x} = (1+x)^{\frac{1}{2}} = \dots = 1 + \frac{1}{2}\,x - \frac{1}{8}\,x^2 + \frac{1}{16}\,x^3 - \frac{5}{128}\,x^4 + \dots$

Hence, $\sqrt{1.1} = \sqrt{1+0.1} = 1 + \frac{1}{2}\,(0.1) - \frac{1}{8}\,(0.1)^2 + \frac{1}{16}\,(0.1)^3 - \frac{5}{128}\,(0.1)^4 + \dots$

Now: $\sqrt{1.1} = 1 + \frac{1}{2}\,(0.1) - \frac{1}{8}\,(0.1)^2 \pm \frac{1}{16}\,(0.1)^3 = 1.04875 \pm 0.0000625$

EXAMPLE 8: The Lorentz factor from Special Relativity is: $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$

Here, v is the velocity of an object and c is the speed of light. Since most objects travel at well below the speed of light, $v << c$ so $\frac{v}{c} << 1$ which means $\frac{v^2}{c^2} \approx 0$. Hence, the Binomial Series can used to approximate γ:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \left(1 + \left(-\frac{v^2}{c^2}\right)\right)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) + \dots \approx 1 + \frac{v^2}{2\,c^2}$$

Indeed, you'll see the approximation $\gamma \approx 1 + \frac{v^2}{2\,c^2}$ when simplifying equations in advanced physics courses!

HOMEWORK: Section 11.3: 9 - 65 every other odd, 90*; Section 11.4: 9 - 65 every other odd, 79*, 81*